Student Workbook

Exponential and Logarithmic Models

Name: ______________________ Date: ____________ Class: ________

ESSENTIAL QUESTION

How do exponential and logistic models let us describe real-world growth, decay, and cooling — and how do we work backward from data to find the rate constant k?

Lesson Overview

Many real quantities change at a rate proportional to their current value, giving the continuous exponential model $A(t) = A0 \cdot e^{kt}$: growth when $k>0$, decay when $k<0$. Special cases include continuous compound interest, half-life, and doubling time. When growth is limited by a maximum carrying capacity, we instead use the logistic model. Newton's Law of Cooling describes an object's temperature approaching room temperature.

$$A(t) = A0 \cdot e^{kt}$$

$$f(t) = c / (1 + a \cdot e^{-bt})$$

$$T(t) = Ts + (T0 - Ts) \cdot e^{kt}$$

Exp.	A0 = initial amount, k = rate constant (growth $k>0$, decay $k<0$)
Logistic	c = carrying capacity, $a,b>0$ shape constants — bounded growth
Cooling	Ts = surrounding temp, T0 = initial temp — approaches Ts as $t \to \infty$

Key Formulas

- Half-life: $t(1/2) = \ln(2) / |k|$. Doubling time: $t(2) = \ln(2) / k$
- Continuous compound interest: $A = P \cdot e^{rt}$ (P = principal, r = annual rate, t = years)
- To find k from data, isolate e^{kt}, take ln of both sides, then divide by t.

Worked Examples

EXAMPLE 1

A culture of 500 bacteria doubles every 3 hours. Find the population after 10 hours.

- Doubling model: $A(t) = 500 \cdot e^{kt}$ where $e^{3k}=2$, so $k = \ln(2)/3 \approx 0.2310$
- $A(10) = 500 \cdot e^{0.2310 \cdot 10} = 500 \cdot e^{2.310}$
- $A(10) = 500 \cdot (10.08) \approx 5039$ bacteria

Answer: A(10) ≈ 5039 bacteria

EXAMPLE 2

A fossil contains 30% of its original carbon-14 (half-life 5730 yr). Estimate its age.

- $k = -\ln(2)/5730 \approx -0.00012097$
- $0.30 = e^{kt} \to t = \ln(0.30)/k$
- $t = (-1.204)/(-0.00012097) \approx 9954$ years

Answer: $t \approx 9950$ years

EXAMPLE 3

$8000 is invested at 4.5% compounded continuously. How long until it doubles?

- $A = P \cdot e^{rt} \to 2P = P \cdot e^{0.045t} \to 2 = e^{0.045t}$
- Take ln of both sides: $\ln(2) = 0.045t$
- $t = \ln(2)/0.045 \approx 15.4$ years

Answer: $t \approx 15.4$ years

EXAMPLE 4

Coffee cools from 90°C to 70°C in 5 min in a 22°C room. Find T(20) (Newton's cooling).

- $T(t) = 22 + 68 \cdot e^{kt}$. $T(5)=70$: $48 = 68 \cdot e^{5k} \to k = \ln(48/68)/5 \approx -0.0697$
- $T(20) = 22 + 68 \cdot e^{-0.0697 \cdot 20} = 22 + 68 \cdot e^{-1.393}$
- $T(20) = 22 + 68 \cdot (0.2483) \approx 38.9$°C

Answer: $T(20) \approx 39$°C

EXAMPLE 5

A logistic model is $f(t) = 50000 / (1 + 24 \cdot e^{-0.15t})$. Find f(20) and the carrying capacity.

- Carrying capacity = numerator's limit as $t \to \infty$ = 50000
- $f(20) = 50000 / (1 + 24 \cdot e^{-3}) = 50000 / (1 + 24 \cdot 0.04979)$
- $f(20) = 50000 / 2.195 \approx 22{,}780$

Answer: $f(20) \approx 22{,}780$; carrying capacity = 50,000

Guided Practice

Work through each problem below. Use the hint if you get stuck — write your final answer on the last line.

1 A rabbit population grows from 2000 to 3500 in 4 years. Predict the population at t=10 years.

Hint: Find k from $A0 \cdot e^{4k}=3500$ with A0=2000. Then evaluate $A(10)=2000 \cdot e^{10k}$.

2 A sample decays from 80 g to 50 g in 12 years. Find the half-life.

Hint: First find k from $50=80 \cdot e^{12k}$, then use $t(1/2)=\ln(2)/|k|$.

3 \$5000 is invested at 3.2% compounded continuously for 8 years. Find the balance.

Hint: Use $A=P\cdot e^{rt}$ with P=5000, r=0.032, t=8.

4 A turkey cools from 165°F to 140°F in 30 min in a 70°F kitchen. Find T(90).

Hint: Use $T(t)=70+95\cdot e^{kt}$. Find k from T(30)=140, then evaluate T(90).

5 A logistic model is $P(t)=8000/(1+15\cdot e^{-0.2t})$. Find P(0) and P(10).

Hint: Substitute t=0 and t=10 directly into the formula.

Key Vocabulary

Exponential Growth Model	$A(t) = A0\cdot e^{kt}$ with $k > 0$ — quantity increases at a rate proportional to its size.
Exponential Decay Model	$A(t) = A0\cdot e^{kt}$ with $k < 0$ — quantity decreases at a rate proportional to its size.
Continuous Compound Interest	$A = P\cdot e^{rt}$ — interest compounded an infinite number of times per year.
Half-Life	The time required for a decaying quantity to reduce to half its value: $t(1/2) = \ln(2)/\|k\|$.
Doubling Time	The time required for a growing quantity to double: $t(2) = \ln(2)/k$.
Logistic Growth Model	$f(t) = c/(1+a\cdot e^{-bt})$ — growth that slows as it approaches a carrying capacity.
Carrying Capacity	The maximum sustainable value c that a logistic model approaches as $t\to\infty$.
Newton's Law of Cooling	$T(t) = Ts+(T0-Ts)e^{kt}$ — an object's temperature approaches ambient temperature Ts over time.

Check Your Understanding

Circle the letter of the correct answer for each question.

1 **$A(t) = A0\cdot e^{kt}$ with $k > 0$ describes:** Multiple Choice

A Exponential decay

B Exponential growth

C A constant quantity

D A logistic curve

2 **The time for a quantity to fall to half its value is called the:** Multiple Choice

- **A** Doubling time
- **B** Half-life
- **C** Decay constant
- **D** Carrying capacity

3 **In $A = P\cdot e^{rt}$, what does r represent?** Multiple Choice

- **A** The principal
- **B** The elapsed time
- **C** The annual interest rate
- **D** The final balance

4 **In $f(t) = c/(1+a\cdot e^{-bt})$, what does c represent?** Multiple Choice

- **A** The growth rate
- **B** The initial value
- **C** The carrying capacity
- **D** The time constant

5 **Newton's Law of Cooling says an object's temperature approaches:** Multiple Choice

- **A** 0°C
- **B** The surrounding (ambient) temperature
- **C** Its initial temperature
- **D** Infinity

Common Mistakes to Avoid

✗ Confusing the carrying capacity c with the initial value of a logistic model.	✓ c is the horizontal asymptote as t→∞ — the LARGEST value, not f(0).
✗ Forgetting the negative sign on k when modeling decay or cooling toward a lower temperature.	✓ If the quantity is decreasing, k must be negative — check the sign after solving for k.
✗ Using t(1/2)=ln(2)/k without the absolute value, giving a negative half-life.	✓ Half-life and doubling time are always positive: t(1/2)=ln(2)/|k|.
✗ Plugging into Newton's Law of Cooling without subtracting the surrounding temperature first.	✓ Always find k from (T0−Ts) and the observed temperature drop, not from T0 alone.

Math Tips

★ To find k from two data points, isolate e^{kt}, take ln of both sides, then divide by t.

★ Carrying capacity = the numerator of a logistic model; it's also the limit as $t \to \infty$ (denominator → 1).

★ For cooling/warming problems, always work with $(T - T_s)$, the difference from ambient temperature.

★ Continuous interest (e^{rt}) is a special case of exponential growth — compare it to the general model.

★ Round k to at least 4-5 decimal places before using it in further calculations to avoid rounding error.